

Nonlinear integral equations for finite volume excited state energies of the $O(3)$ and $O(4)$ nonlinear σ -models

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Abstract

We propose nonlinear integral equations for the finite volume one-particle energies in the $O(3)$ and $O(4)$ nonlinear σ -models. The equations are written in terms of a finite number of components and are therefore easier to solve numerically than the infinite component excited state TBA equations proposed earlier. Results of numerical calculations based on the nonlinear integral equations and the excited state TBA equations agree within numerical precision.

1 Introduction

A better theoretical understanding of finite size (FS) effects is one of the most important problems in Quantum Field Theory (QFT). The study of FS effects is a useful method of analysing the structure of QFT models and it is an indispensable tool in the numerical simulation of lattice field theories.

Finite size effects can be studied through the volume dependence of the mass gap of the theory, the usefulness of which is demonstrated [1] by the introduction of the Lüscher-Weisz-Wolff (LWW) running coupling that enables the interpolation between the large volume (non-perturbative) and the small volume (perturbative) regions in both two-dimensional sigma models and QCD.

The study of the Lüscher-Weisz-Wolff running coupling is useful in the two-dimensional $O(N)$ nonlinear sigma (NLS) models, because according to recently performed high precision Monte Carlo measurements of the LWW running coupling [2] the cutoff effects look linear in these models, in contrast to perturbative considerations. The knowledge of the exact value of the LWW coupling enables one to make better fits for the cutoff effects, and thus to determine more accurately the functional form of the lattice artifacts.

Our aim in this paper is to propose nonlinear integral equations (NLIEs) for the one-particle states of the $O(3)$ and $O(4)$ NLS models that allow for a fast and accurate numerical calculation of the LWW coupling. Although this calculation has recently been done in [3] using the excited state Thermodynamic Bethe Ansatz (TBA) technique [4, 5, 6, 7, 8, 9], the difficulties corresponding to the infinite number of components in the TBA equations make the numerical calculations slow and at the same time restrict their accuracy too. Therefore it is desirable to work with the more convenient NLIE technique.

Another point of the construction of these excited state NLIEs is to demonstrate that, similarly to the case of the sine-Gordon model [10, 11], the NLIE technique can be extended to describe the finite size excited states also in the family of NLS models.

The infinite set of TBA equations for the ground state of the $O(3)$ and $O(4)$ NLS models were given in [12, 13] and [14, 15] respectively. The derivation of the equations were based on the fact that NLS models can be represented as (limits of)

certain perturbed conformal field theories [12, 13, 14, 15].

The TBA description of the excited states is less systematic in the continuum models as it is for the ground state problem. Although a lot of different methods have been worked out to obtain excited state TBA equations in different models [4, 5, 6, 7, 8, 9], a general construction has not discovered, yet. The generalization of the TBA equations of the $O(3)$ and $O(4)$ NLS models to one-particle excited states was proposed recently [3]. The sigma-model TBA equations consist of infinitely many components, which makes their numerical analysis difficult.

The NLIEs for the ground state of the $O(3)$ NLS model were proposed in [16], based on the statement, according to which the $O(3)$ NLS model can be expressed as a certain limit of appropriately perturbed Z_N parafermion conformal field theories [12].

The ground state NLIEs for the $O(4)$ NLS model were derived in [17], using the integrable lattice regularization of the model [20, 21, 22].

Our main purpose in this paper to propose excited state NLIEs for the one-particle states of the $O(3)$ and $O(4)$ NLS models. This is achieved in sections 4 and 5 using the assumption that the excited state NLIEs differ from the ground state ones only in additional source terms, and if it is necessary in additional quantization conditions.

The paper is organised as follows. In section 2 we recall the TBA integral equations, Y-systems and NLIEs corresponding to the ground state problem. In section 3 we briefly summarize the one-particle TBA equations of the models. In sections 4 and 5 we propose excited state NLIEs for the $O(4)$ and $O(3)$ NLS models respectively. Numerical solutions of the NLIEs and their comparison to perturbation theory and to TBA results are discussed in section 6. Finally our conclusions are summarized in section 7.

2 The ground state problem of the $O(4)$ and $O(3)$ NLS models (TBA and NLIE)

In this section we give a short review of the ground state TBA equations, Y-systems, and nonlinear integral equations (NLIEs) for the $O(4)$ and $O(3)$ NLS models.

The ground state energy of a two-dimensional integrable model enclosed in a finite box with periodic boundary conditions can be determined by the solutions of the TBA integral equations [18]. The TBA equations of the $O(4)$ and $O(3)$ NLS models can be encoded in infinite Dynkin-diagrams. (See fig. 1a and 1b)



Figure 1: Dynkin-diagrams associated with the Y-systems of the a.) $O(4)$ and b.) $O(3)$ σ -models.

The unknown functions y_a are associated to nodes of the Dynkin-diagram and the TBA equations are of the form [12, 13, 14, 15]

$$y_a(x) = \exp \left\{ \delta_{a1} \mathcal{D}(x) + \sum_{b=1}^{\infty} I_{ab} (K * \log Y_b)(x) \right\}, \quad a = 1, 2, \dots \quad (1)$$

where

$$Y_a(x) = 1 + y_a(x), \quad K(x) = \frac{1}{4 \cosh \frac{\pi}{2} x}, \quad \mathcal{D}(x) = -ml \cosh \left(\frac{\pi}{2} x \right), \quad (2)$$

the $*$ denotes the convolution, i.e. $(f * g)(x) = \int_{-\infty}^{\infty} dy f(x-y)g(y)$, m is the mass gap in infinite volume, l is the box size and I_{ab} is the incidence matrix of the Dynkin-diagram. The TBA equations of the $O(4)$ NLS model correspond to the diagram shown in figure 1a [14, 15], where the oriented double line at the beginning of the diagram means

$$I_{12} = 2, \quad I_{21} = 1. \quad (3)$$

The TBA equations of the $O(3)$ NLS model are encoded into a D_{∞} diagram shown in figure 1b. The ground state energy can be calculated from the solutions of the TBA equations [12, 13, 14, 15]

$$E_0(l) = -\frac{m}{4} \int_{-\infty}^{\infty} dx \cosh \left(\frac{\pi x}{2} \right) \log Y_1(x). \quad (4)$$

The solutions of the TBA equations also satisfy the so called Y-system equations [19]

$$y_a(x+i) y_a(x-i) = \prod_b Y_b(x)^{I_{ab}}. \quad (5)$$



Figure 2: Graphical representation for the NLIEs of the a.) $O(4)$ and b.) $O(3)$ σ -models.

There is another alternative to calculate the ground state energy of the $O(4)$ NLS model, namely the nonlinear integral equation technique. The nonlinear integral equations for the ground state of the $O(4)$ NLS were derived in [17] using the light-cone lattice approach of the model [20]. The nonlinear integral equations in this case are of the form

$$\begin{aligned}
\log y_1(x) &= \mathcal{D}(x) + 2(K^{+\gamma} * \log U)(x) + 2(K^{-\gamma} * \log \bar{U})(x), \\
\log a(x) &= (F * \log U)(x) - (F^{+2(1-\gamma)} * \log \bar{U})(x) \\
&\quad + (K^{-\gamma} * \log Y_1)(x), \\
\log \bar{a}(x) &= (F * \log \bar{U})(x) - (F^{-2(1-\gamma)} * \log U)(x) \\
&\quad + (K^{+\gamma} * \log Y_1)(x), \\
U(x) &= 1 + a(x), \quad \bar{U}(x) = 1 + \bar{a}(x), \quad Y_1(x) = 1 + y_1(x),
\end{aligned} \tag{6}$$

where $0 < \gamma < 1/2$ is an arbitrary fixed parameter,

$$F(x) = \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{-|q|-iqx}}{2 \cosh(q)}, \tag{7}$$

and we have used the notation

$$f^{\pm\eta}(x) = f(x \pm i\eta). \tag{8}$$

In this case the form of the ground state energy is the same as it is in the TBA case (4). Equations (6) contain only three real unknown functions because $y_1(x)$ and $Y_1(x)$ are real and $\bar{a}(x)$ is the complex conjugate of $a(x)$, therefore they serve an efficient basis for numerical calculations. The graphical notation of these equations is depicted in figure 2a. The big “bubble” denotes the complex auxiliary functions which resum the contributions of those TBA nodes, which are inside it. In our

notation the names of the complex unknown functions and the kernel function are indicated.

In [17] the eqs. (6) were derived on a Bethe Ansatz solvable lattice and certainly the TBA equations of the model can be also derived from the Bethe Ansatz solution of the model. Thus, it turns out that the function $y_1(x)$ of eqs. (6) are exactly the same as the function $y_1(x)$ of the TBA equations (1). Furthermore the connection between the complex $a(x)$ and $\bar{a}(x)$ variables and the TBA variables can be expressed by the following formula

$$U(x + i\gamma) \bar{U}(x - i\gamma) = Y_2(x), \quad (9)$$

which is a very important formula from the NLIE technique point of view, because this relation allows one to reduce the infinite component TBA system to a finite component NLIEs.

NLIEs are also available for the ground state of the $O(3)$ NLS model. In [16] NLIEs were proposed to a class of perturbed parafermion conformal field theories, which reduce to the $O(3)$ NLS model in a certain limit [12]. Taking the appropriate limit the NLIEs take the form

$$\begin{aligned} \log y_1(x) &= \mathcal{D}(x) + \log y_2(x), \\ \log y_2(x) &= (K^{+\gamma} * \log U)(x) + (K^{-\gamma} * \log \bar{U})(x), \\ \log a(x) &= (F * \log U)(x) - (F^{+2(1-\gamma)} * \log \bar{U})(x) \\ &\quad + (K^{-\gamma} * \log Y_1)(x) + (K^{-\gamma} * \log Y_2)(x), \\ \log \bar{a}(x) &= (F * \log \bar{U})(x) - (F^{-2(1-\gamma)} * \log U)(x) \\ &\quad + (K^{+\gamma} * \log Y_1)(x) + (K^{+\gamma} * \log Y_2)(x), \\ U(x) &= 1 + a(x), \quad \bar{U}(x) = 1 + \bar{a}(x), \quad Y_a(x) = 1 + y_a(x), \quad a = 1, 2 \end{aligned} \quad (10)$$

where $0 < \gamma < 1/2$ is an arbitrary fixed parameter, the kernel and source functions are the same as in (2),(7) and the ground state energy is given by the formula (4) as in the TBA case. The graphical representation of these equations is given in figure 2b.

Although these equations are only conjectured ones, one can recognise that the functions $y_1(x)$ and $y_2(x)$ are the same as the ones of the corresponding TBA equations (1), $a(x)$ and $\bar{a}(x)$ are the complex conjugates of each other and the connection

between the TBA and NLIE variables can be expressed (similarly to the $O(4)$ case) by the formula

$$U(x + i\gamma) \bar{U}(x - i\gamma) = Y_3(x), \quad (11)$$

which somehow ensures the reduction of the infinite component Y-system.

3 TBA equations for the one-particle states of the $O(4)$ and $O(3)$ NLS models

In this section we briefly review the first excited states TBA equations of the $O(4)$ and $O(3)$ NLS models proposed in [3].

In [3] the main assumption was (based on previous experience with the sine-Gordon case) that the Y-system (5) describes not only the ground state of the model, but remains valid also for the first excited state. Then Lüscher's mass gap formula [23], valid for asymptotically large volumes, was used to determine the infinite volume solution of the excited state Y-system, which is sufficient to derive the one-particle excited state TBA equations. It is important from the NLIE technique point of view that the infinite volume solutions of the Y-system can be expressed by an infinite volume t-system, which is of the form

$$t_p(x + i) t_p(x - i) = B(x - ip) B(x + ip) + t_{p-1}(x) t_{p+1}(x), \quad p = 1, 2, \dots \quad (12)$$

where

$$t_0(x) = 0, \quad (13)$$

$$B(x) = x, \quad \text{in the } O(4) \text{ case, and} \quad (14)$$

$$B(x) = x^2 + 1, \quad \text{in the } O(3) \text{ case.} \quad (15)$$

The solution of (12) can be represented in the following form

$$t_p(x) = \sum_{l=1}^p \lambda_l^{(p)}(x), \quad \lambda_l^{(p)}(x) = B[x + i(2l - p - 1)]. \quad (16)$$

Using the infinite volume t-system (12), the infinite volume solutions of the Y-system can be given as

$$y_p^\infty(x) = \frac{t_{p-1}(x) t_{p+1}(x)}{B(x - ip) B(x + ip)}, \quad Y_p^\infty(x) = \frac{t_p(x - i) t_p(x + i)}{B(x - ip) B(x + ip)}, \quad p = 1, 2, \dots \quad (17)$$

The infinite volume Y-system of the $O(3)$ and $O(4)$ NLS models [3] are of the form

$$y_p^\infty(x+i) y_p^\infty(x-i) = Y_{p-1}^\infty(x) Y_{p+1}^\infty(x) \quad p = 2, 3, \dots \quad (18)$$

where $y_1^\infty(x) = 0$, because the Y-system element corresponding to the massive node of the Dynkin-diagram tends to zero in the infinite volume limit. The usefulness of these infinite volume solutions is that, they give the qualitative position of the zeroes of the $y_a(x)$ functions of the Y-system, which is sufficient [3, 24] for the derivation of excited state the TBA integral equations. Thus the one particle TBA equations of the $O(4)$ NLS model are of the form [3]

$$y_a(x) = \tau^2(x) \exp \left\{ \delta_{a1} \mathcal{D}(x) + \sum_{b=1}^{\infty} I_{ab} (K * \log Y_b)(x) \right\}, \quad a = 1, 2, \dots \quad (19)$$

where

$$\tau(x) = \tanh \left(\frac{\pi x}{4} \right), \quad (20)$$

and I_{ab} the incidence matrix of figure 1a. The energy of this state takes the form

$$E_1(l) = m - \frac{m}{4} \int_{-\infty}^{\infty} dx \cosh \left(\frac{\pi x}{2} \right) \log Y_1(x). \quad (21)$$

Similarly the one particle TBA equations of the $O(3)$ NLS model can be read off from [3] and take the form

$$\begin{aligned} y_1(x) &= e^{\mathcal{D}(x)} y_2(x), \\ y_2(x) &= \tau(x-h_3) \tau(x+h_3) \exp \{ (K * \log Y_3)(x) \} \\ y_a(x) &= \tau(x-h_{a-1}) \tau(x+h_{a-1}) \tau(x-h_{a+1}) \tau(x+h_{a+1}) e^{\left\{ \sum_{b=1}^{\infty} I_{ab} (K * \log Y_b)(x) \right\}} \\ a &= 3, 4, \dots \end{aligned} \quad (22)$$

and in addition to these integral equations one has to impose quantization conditions for the h_a zeroes of the $y_a(x)$ functions to ensure that $y_a(\pm h_a \pm i) = -1$, which is the consequence of the (5) Y-system relations. These quantization equations can be obtained by the analytic continuation of (22) and are of the form

$$\gamma(h_s - h_{s-1}) + \gamma(h_s + h_{s-1}) + \gamma(h_s - h_{s+1}) + \gamma(h_s + h_{s+1}) - \frac{1}{i} \sum_{b=1}^{\infty} I_{sb} (K^{-1} * \log Y_b)(x) = \pi, \quad (23)$$

for $s = 3, 4, \dots$ where $h_2 = 0$, $\gamma(x) = 2 \arctan \tau(x)$ and

$$(K^{-1} * \log Y_b)(x) = i \mathcal{P} \int_{-\infty}^{\infty} \frac{dy}{4} \frac{\log Y_b(y)}{\sinh \frac{\pi}{2}(x-y)}, \quad (24)$$

is a principal value integration.

Our main goal in this paper to propose NLIEs equivalent to these infinite component excited state TBA equations.

4 Nonlinear integral equations for the first excited state of the $O(4)$ NLS model

In this section we propose nonlinear integral equations for the first excited state of the $O(4)$ NLS model. The proposal is based on the assumption that the excited states NLIEs differ from the ground state ones only in additional source terms, plus if it is necessary in quantization conditions. In addition we assume that the functional form of these additional source terms does not depend directly on the volume. They can depend on the volume only through some complex objects, on which we impose quantization conditions. Such objects may be the zeroes of the Y-system elements, or some roots of the Bethe Ansatz equations as it happens in the sine-Gordon case [11]. These assumptions are valid for the NLIEs in the sine-Gordon model, where the form of the excited state NLIEs of ref. [11] differ from the ground state NLIE of ref. [10] only in additional source terms and in quantization conditions.

In this model one can even give a physical interpretation of the various terms of the equation. The integral term represents the contribution of the infinite number of real Bethe Ansatz roots. The presence of these roots can be regarded as filling the Dirac sea and thus the presence of this term in the NLIE of the sine-Gordon model is somehow related to the vacuum structure of the theory. For an excited state of the model one has to consider a slightly modified configuration of the Bethe Ansatz roots. It turns out that in the sine-Gordon model these modifications can be taken into account by adding source terms to the NLIE of the ground state problem [11]. Similarly, the excited state TBA equations of the sine-Gordon model differ from the ground state ones only by some source terms [24]. Finally, additional support for

our assumptions comes from a direct calculation in the higher-spin vertex model, which reduce to the $O(4)$ NLS model in the infinite spin limit. These calculations show that excited state NLIEs differ from the ground state ones again only by some source terms, which depend on the volume only through some complex objects, on which quantization conditions have to be imposed. (As an additional example see [28] for the $S=1$ special case.) However, we cannot directly derive the one-particle NLIEs of the $O(4)$ NLS model from the (limit of the) higher-spin vertex model, since the solvable lattice model is describing only a subsector of the Hilbert space of the model [20, 21] and unfortunately the one-particle states are not in this subsector.

Accepting these assumptions our task is to find out the necessary source terms of the excited state NLIEs. This is achieved similarly to the case of the derivation of the excited state TBA equations of the model, where the infinite volume solutions of the equations allowed one to deduce the TBA integral equations [24]. The unknown functions of the excited state problem will be denoted in the same way as in eqs. (6). Assuming that the relation (9) holds also for the first excited states, we can use this relation in the equation for the massive mode in (1). This leads to the following Ansatz for the first excited state NLIE problem:

$$\begin{aligned}
\log y_1(x) &= \mathcal{D}(x) + \log \tau^2(x) + 2(K^{+\gamma} * \log U)(x) + 2(K^{-\gamma} * \log \bar{U})(x), \\
\log a(x) &= \mathcal{F}_a(x) + (F * \log U)(x) - (F^{+2(1-\gamma)} * \log \bar{U})(x) \\
&\quad + (K^{-\gamma} * \log Y_1)(x), \\
\log \bar{a}(x) &= \mathcal{F}_{\bar{a}}(x) + (F * \log \bar{U})(x) - (F^{-2(1-\gamma)} * \log U)(x) \\
&\quad + (K^{+\gamma} * \log Y_1)(x), \\
U(x) &= 1 + a(x), \quad \bar{U}(x) = 1 + \bar{a}(x), \quad Y_1(x) = 1 + y_1(x),
\end{aligned} \tag{25}$$

where $0 < \gamma < 1/2$ is an arbitrary fixed parameter, the energy formula identical to (21) and our task is to determine the presently unknown $\mathcal{F}_a(x)$ and $\mathcal{F}_{\bar{a}}(x)$ source functions. Using the assumption that these functions do not depend on the volume directly, one can determine them from the infinite volume solution of the proposed equations, which can be determined from the solution of the infinite volume t-system (12) and Y-system (18). Using the construction of [17, 25], the infinite volume

solutions of the Ansatz (25) take the form

$$a_\infty(x) = \frac{\lambda_1^{(2)}(x+i-i\gamma)}{\lambda_2^{(2)}(x+i-i\gamma)} = \frac{t_1(x-i\gamma)}{B[x+i(2-\gamma)]} = \frac{x-i\gamma}{x+i(2-\gamma)}, \quad (26)$$

$$\bar{a}_\infty(x) = \frac{\lambda_2^{(2)}(x-i+i\gamma)}{\lambda_1^{(2)}(x-i+i\gamma)} = \frac{t_1(x+i\gamma)}{B[x-i(2-\gamma)]} = \frac{x+i\gamma}{x-i(2-\gamma)}, \quad (27)$$

$$U_\infty(x) = 1 + a_\infty(x) = 2 \frac{x+i(1-\gamma)}{x+i(2-\gamma)}, \quad (28)$$

$$\bar{U}_\infty(x) = 1 + \bar{a}_\infty(x) = 2 \frac{x-i(1-\gamma)}{x-i(2-\gamma)}, \quad (29)$$

and $y_1^\infty(x) = 0$. One can check that $a(x)$ is the complex conjugate of $\bar{a}(x)$ and that these solutions satisfy the relation $U_\infty(x+i\gamma) \bar{U}_\infty(x-i\gamma) = Y_2^\infty(x)$, which is the infinite volume limit of (9). Once one knows the infinite volume solutions of the Ansatz (25), one can substitute these formulae into them, and can compute explicitly the unknown source functions. The most convenient way of this is to take the derivative of eqs. (25) and work in Fourier space, and in the end return to the coordinate space and integrate once. After these simple calculations one gets

$$\mathcal{F}_a(x) = \chi(x+i(1-\gamma)) + \chi_K(x+i(1-\gamma)), \quad (30)$$

$$\mathcal{F}_{\bar{a}}(x) = -\chi(x-i(1-\gamma)) - \chi_K(x-i(1-\gamma)), \quad (31)$$

where

$$\chi(x) = 2\pi i \int_0^x dy F(y) = i \int_{-\infty}^{\infty} \frac{dq}{q} \sin(qx) \frac{e^{-|q|}}{2 \cosh(q)}, \quad (32)$$

$$\chi_K(x) = 2\pi i \int_0^x dy K(y) = i \arctan \sinh\left(\frac{\pi x}{2}\right). \quad (33)$$

Note that these functions are nothing but the odd primitive of the kernel functions of the integral terms of the ground state equations (6), which is similar to what happens for the NLIE of the sine-Gordon model in the case of the one hole excitation [11].

We solved these NLIEs numerically for a number of cases and the results we found always agreed with those obtained from the numerical solution of the TBA

equations (19) (see section 6.). This means that the two methods are equivalent. On the other hand the TBA equations were tested previously by using Lüscher's formula, Monte Carlo measurements [2] and 3-loop perturbation theory [26]. Thus we are confident that eqs. (25) with source terms (30,31) correctly describe the one-particle excited states of the $O(4)$ NLS model. They are superior to the TBA equations since they contain only three real unknown functions leading to faster convergence in numerical calculations.

5 Excited states nonlinear integral equations for the $O(3)$ NLS model

In this section following the method described in the previous section we propose NLIEs for the first excited state of the $O(3)$ NLS model. According to our assumptions the Ansatz for the NLIEs is of the form

$$\begin{aligned}
\log y_1(x) &= \mathcal{D}(x) + \log y_2(x), \\
\log y_2(x) &= \log \tau(x - h_3) + \log \tau(x + h_3) + (K^{+\gamma} * \log U)(x) + (K^{-\gamma} * \log \bar{U})(x), \\
\log a(x) &= \mathcal{F}_a(x) + (F * \log U)(x) - (F^{+2(1-\gamma)} * \log \bar{U})(x) \\
&\quad + (K^{-\gamma} * \log Y_1)(x) + (K^{-\gamma} * \log Y_2)(x), \\
\log \bar{a}(x) &= \mathcal{F}_{\bar{a}}(x) + (F * \log \bar{U})(x) - (F^{-2(1-\gamma)} * \log U)(x) \\
&\quad + (K^{+\gamma} * \log Y_1)(x) + (K^{+\gamma} * \log Y_2)(x), \\
U(x) &= 1 + a(x), \quad \bar{U}(x) = 1 + \bar{a}(x), \quad Y_a(x) = 1 + y_a(x), \quad a = 1, 2
\end{aligned} \tag{34}$$

where $0 < \gamma < 1/2$ is an arbitrary fixed parameter, $y_1(x)$ and $y_2(x)$ are the same variables as the ones in the excited state TBA eqs. (22), and h_3 is the zero of the $y_3(x)$ Y-system element of the excited state problem (i.e. $y_3(\pm h_3 \pm i) = -1$), therefore an additional quantization equation must be imposed on this zero. This quantization condition can also be derived from the infinite volume solution of our equations, and will be discussed at the end of this section. The energy expression in turn is the same as the TBA one (21).

According to our method first we have to determine the infinite volume solution of our Ansatz (34). This can be achieved using the solutions of the infinite vol-

ume t-system (12-16), and applying the construction of refs. [17, 25]. After some straightforward calculations one gets

$$a_\infty(x) = \frac{\lambda_1^{(3)}(x+i-i\gamma) + \lambda_2^{(3)}(x+i-i\gamma)}{\lambda_3^{(3)}(x+i-i\gamma)} = 2 \frac{(x-i\gamma)^2}{(x+i(4-\gamma)) \cdot (x+i(2-\gamma))}, \quad (35)$$

$$\bar{a}_\infty(x) = \frac{\lambda_2^{(3)}(x-i+i\gamma) + \lambda_3^{(3)}(x-i+i\gamma)}{\lambda_1^{(3)}(x-i+i\gamma)} = 2 \frac{(x+i\gamma)^2}{(x-i(4-\gamma)) \cdot (x-i(2-\gamma))}, \quad (36)$$

$$U_\infty(x) = 1 + a_\infty(x) = 3 \frac{(x+h_0+i(1-\gamma)) \cdot (x-h_0+i(1-\gamma))}{(x+i(4-\gamma)) \cdot (x+i(2-\gamma))}, \quad (37)$$

$$\bar{U}_\infty(x) = 1 + \bar{a}_\infty(x) = 3 \frac{(x+h_0-i(1-\gamma)) \cdot (x-h_0-i(1-\gamma))}{(x-i(4-\gamma)) \cdot (x-i(2-\gamma))}, \quad (38)$$

$$y_2^\infty(x) = \frac{t_1(x) t_3(x)}{B(x+2i) B(x-2i)} = 3 \frac{(x+h_0) (x-h_0)}{(x+3i) (x-3i)}, \quad (39)$$

$$Y_2^\infty(x) = 1 + y_2^\infty(x) = \frac{t_2(x+i) t_2(x-i)}{B(x+2i) B(x-2i)} = 3 \frac{(x+i) (x-i)}{(x+3i) (x-3i)}, \quad (40)$$

where $h_0 = \lim_{l \rightarrow \infty} h_3 = \sqrt{5/3}$, the infinite volume limit of the zero h_3 , and certainly $y_1(x) = 0$ in the infinite volume limit. The infinite volume solutions (35-40) also satisfy the important relation $U_\infty(x+i\gamma)U_\infty(x-i\gamma) = Y_3^\infty(x)$, which allows one to cut the infinite TBA equations.

Having the infinite volume solutions of our Ansatz, using the method detailed in the previous section one can calculate the unknown source terms (at least their infinite volume limit). After simple calculations one gets

$$\mathcal{F}_a(x) = \chi(x+h_0+i(1-\gamma)) + \chi(x-h_0+i(1-\gamma)) + 2\chi_K(x+i(1-\gamma)), \quad (41)$$

$$\mathcal{F}_{\bar{a}}(x) = -\chi(x+h_0-i(1-\gamma)) - \chi(x-h_0-i(1-\gamma)) - 2\chi_K(x-i(1-\gamma)), \quad (42)$$

according to our assumption these source terms may depend on the volume through the objects, which can be found in their arguments, namely through the volume

dependence of h_0 . Therefore we replace h_0 by its finite volume value h_3 in (41,42), and according to our conjecture that will be the form of the source functions for finite volume,

$$\mathcal{F}_a(x) = \chi(x + h_3 + i(1 - \gamma)) + \chi(x - h_3 + i(1 - \gamma)) + 2\chi_K(x + i(1 - \gamma)), \quad (43)$$

$$\mathcal{F}_{\bar{a}}(x) = -\chi(x + h_3 - i(1 - \gamma)) - \chi(x - h_3 - i(1 - \gamma)) - 2\chi_K(x - i(1 - \gamma)). \quad (44)$$

The eqs. (34) must be supplemented by the quantization condition of the zero h_3 , similarly as it was in the TBA case. The quantization condition can be found out from the infinite volume solutions (35-40). One can see from these solutions that $U_\infty(\pm h_0 - i(1 - \gamma)) = 0$. Assuming that this relation remains valid also for finite volume, one can infer that $U(\pm h_3 - i(1 - \gamma)) = 0$, from which follows that

$$a(h_3 - i(1 - \gamma)) = -1. \quad (45)$$

Taking the logarithm of this, and using eqs. (34) with (43,44), one gets the quantization condition for h_3

$$\mathcal{A}(h_3) = \pi, \quad (46)$$

where

$$\begin{aligned} \mathcal{A}(x) &= \frac{1}{i} \chi(x - h_3) + \frac{1}{i} \chi(x + h_3) + \frac{2}{i} \chi_K(x) + \frac{1}{i} (K^{-1} * \log Y_1)(x) \\ &+ \frac{1}{i} (K^{-1} * \log Y_2)(x) + \frac{1}{i} (F^{-(1-\gamma)} * \log U)(x) - \frac{1}{i} (F^{+(1-\gamma)} * \log \bar{U})(x). \end{aligned} \quad (47)$$

To summarize the eqs. (34) with the source functions (43,44), and supplemented by the quantization conditions (46,47) make up the one-particle NLIEs of the $O(3)$ NLS model. In these equations one can also see that the source functions are nothing but the odd primitives of the kernel functions, as in the $O(4)$ case. Nevertheless in this case the situation is a bit more complicated, because they contain some zeroes in their argument, namely the function $\chi(x)$ contains the zero h_3 , which corresponds to the zero of the $y_3(x)$ (in the $y_3(\pm h_3 \pm i) = -1$ sense), and the function $\chi_K(x)$ contains the zero $h_2 = 0$, which corresponds to the zero of the $y_2(x)$ (in the $y_2(\pm h_2 \pm i) = -1$ sense), but for this object we do not need to impose quantization equation because of symmetry reasons.

Following from the construction of these equations, one can analytically verify by solving iteratively these equations for large l that they give back correctly the same leading order correction to the infinite volume mass gap as it is predicted by the Lüscher's formula [23]. For not very large volumes we tested these NLIEs through numerical calculations, and we found that these equations give the same numerical results as the earlier proposed excited state TBA equations (22-24) (see section 6.). Because the corresponding TBA system was tested by the results of 3-loop order perturbation theory [26] and lattice Monte Carlo measurements [2], thus our new excited state NLIEs are also consistent with these methods, and correctly describe the first excited state energy of the $O(3)$ NLS model.

6 Numerical results

In this section we perform numerical checks on our conjectured one-particle NLIEs. We solved numerically both the TBA equations and the NLIEs for the ground state and for the one-particle states at some values of the volume, and we compared the numerical results of the two different methods. Moreover we solved our conjectured NLIEs in the deep ultraviolet region so as to be able to compare our numerical results to the predictions of the asymptotically free perturbation theory.

The numerical method used for solving the TBA equations is described in [3]. The numerical solution of the corresponding NLIEs is rather similar. Namely, we solve numerically the NLIEs by iteration. The starting point for the iteration is the $l \rightarrow \infty$ solution of the equations and the procedure converges rapidly. The one-particle state problem for the $O(3)$ NLS model is more involved since here one step of the iteration includes the calculation of the integrals occurring in (34) together with the calculation of the zero (h_3) from the quantization condition (46,47). Again, the starting point of the iteration procedure is given by the $l \rightarrow \infty$ solution, both for the unknown functions and for the position of the zero h_3 .

We used Simpson's formula, and a cutoff Λ for the numerical evaluations of the integrals running from $-\infty$ to $+\infty$, in such a way, that in the region of $|x| > \Lambda$, the unknown functions are approximated by their infinite volume limit. The magnitude of the cutoff Λ at the values of the volume under consideration has to be 100 in the $O(4)$ case and 150 in the $O(3)$ case, so that the 9-digit numerical precision could be

ml	E_0 (NLIE)	E_0 (TBA)
2	-0.1620289681(1)	-0.16202897(1)
1	-0.6437745719(1)	-0.6437746(1)
10^{-1}	-11.273364587(1)	-11.273364(1)
10^{-2}	-127.22634373(1)	-127.2263(1)
10^{-3}	-1343.4090793(1)	-1343.409(1)
10^{-4}	-13865.238816(1)	
10^{-5}	-141563.8217(1)	
10^{-6}	-1436683.423(1)	

Table 1: NLIE and TBA results for the ground state energy in the $O(4)$ NLS model.

ml	E_0 (NLIE)	E_0 (TBA)
2	-0.1228466915(1)	-0.1228466(1)
1	-0.4862495672(1)	-0.4862496(1)
10^{-1}	-8.006985662(1)	-8.006985(1)
10^{-2}	-87.63570019(1)	-87.6357(1)
10^{-3}	-913.9547387(1)	-913.954(1)
10^{-4}	-9374.188294(1)	

Table 2: NLIE and TBA results for the ground state energy in the $O(3)$ NLS model.

reached. Our numerical results are summarized in Tables 1-4.

One can see from these numerical data, that the numerical results served by our NLIEs agree with those of the TBA equations within the numerical precision.

Now we are able to compare our numerical results to those of the asymptotically free perturbation theory. Having the numerical values of the ground state energies and the one-particle state energies, we can compute numerically the dimensionless finite volume mass gap (LWW coupling)

$$z(ml) = l \cdot [E_1(l) - E_0(l)], \quad (48)$$

for which perturbative results are also available up to 3-loop order [26]. The perturbative formulae necessary to compute the 3-loop perturbative mass gap can be found

ml	E_1 (NLIE)	E_1 (TBA)
2	0.9923340593(1)	0.99233406(1)
1	0.9383970591(1)	0.93839706(1)
10^{-1}	-3.004108884(1)	-3.0041089(1)
10^{-2}	-69.83802786(1)	-69.838028(1)
10^{-3}	-901.2815867(1)	-901.28159(1)
10^{-4}	-10260.214298(1)	
10^{-5}	-111091.0324(1)	
10^{-6}	-1172575.496(1)	

Table 3: NLIE and TBA results for the one-particle state energy in the $O(4)$ NLS model.

ml	E_1 (NLIE)	E_1 (TBA)
2	1.02169721(1)	1.0216972(1)
1	1.084208673(1)	1.084208(1)
10^{-1}	0.77721084(1)	0.77718(1)
10^{-2}	-23.6407101(1)	-23.643(1)
10^{-3}	-406.195912(1)	-406.23(1)
10^{-4}	-5150.21619(1)	

Table 4: NLIE and TBA results for the one-particle state energy in the $O(3)$ NLS model.

ml	$z(l)$ (NLIE)	3-loop PT	2-loop PT
10^{-1}	0.826925570(1)	0.826130	0.8252260
10^{-2}	0.573883159(1)	0.5737662	0.5735488
10^{-3}	0.442127493(1)	0.4420969	0.4420193
10^{-4}	0.360502452(1)	0.3604916	0.3604571
10^{-5}	0.30472789(1)	0.3047233	0.3047056
10^{-6}	0.26410793(1)	0.2641057	0.2640957

Table 5: NLIE and PT results for the finite volume mass gap $z(ml)$ in the $O(4)$ NLS model.

ml	$z(l)$ (NLIE)	3-loop PT	2-loop PT
10^{-1}	0.878419650(1)	0.876058	0.873458
10^{-2}	0.639949901(1)	0.639645	0.638874
10^{-3}	0.507758827(1)	0.507669	0.507358
10^{-4}	0.422397210(1)	0.422363	0.422212

Table 6: NLIE and PT results for the finite volume mass gap $z(ml)$ in the $O(3)$ NLS model.

in [3]. The comparison of our numerical results to the predictions of the perturbation theory can be found in Tables 5 and 6. We listed both the 2-loop and 3-loop perturbative results so that one can infer to the accuracy of the 3-loop perturbation theory at the scales under investigation. From this comparison, at very small ml values one can experience a very nice 4-digit agreement in the $O(3)$ case, and an almost 6-digit agreement in the $O(4)$ case, which is non-trivial, since the conjecture of our NLIEs was based on the large volume asymptotics of the unknown functions. This perfect agreement makes us confident that our conjectured one-particle NLIEs describe the exact finite volume one-particle energies in the $O(3)$ and $O(4)$ NLS models.

7 Summary and Conclusions

In this paper we proposed NLIEs for the one-particle states in the $O(3)$ and $O(4)$ NLS models. The form of these excited state NLIEs are based on the assumption that they differ from the ground state ones only by some source terms, which may depend on the volume through some objects in their argument, on which extra quantization conditions must be imposed. This assumption is mainly motivated by the form of the NLIEs in the sine-Gordon model and by direct calculations in the higher-spin vertex model (which go to the $O(4)$ NLS model in the infinite spin limit). Accepting these assumptions and starting from the explicit infinite volume solution of the first excited state Y-system of the models we were able to find the infinite volume limit of the conjectured equations. This is sufficient to determine the source terms and the quantization conditions. It is interesting to note that in all cases the source terms are the odd primitives of the kernels occurring in the integral terms of the equations, just as in the case of the NLIE in the sine-Gordon model. We have solved these equations numerically and found that results agree with those obtained previously from numerical solution of the excited state TBA equations and in the deep ultraviolet regime the numerical results also agree with the predictions of the 3-loop perturbation theory. This agreement is convincing evidence for the correctness of our integral equations. The advantage of using these NLIEs (instead of the infinite set of TBA equations) is that here the number of unknown function is finite (and small).

An interesting generalization of our results would be to propose excited state NLIEs for the $\phi_{(id,id,adj)}$ perturbation of the $SU(2)_L \times SU(2)_K/SU(2)_{L+K}$ models, which were investigated at rational level K in [16].

It would also be interesting to extend the NLIE technique for all the excited states of the $O(3)$ and $O(4)$ NLS models and for such more complicated TBA systems which can be encoded into the products of two Dynkin-diagrams [27].

Acknowledgments

I would like to thank János Balog for useful discussions and for the critical reading of the manuscript. The author acknowledges the financial support provided through the European Community's Human Potential Programme under contract HPRN-

CT-2002-00325, 'EUCLID'. This investigation was also supported in part by the Hungarian National Science Fund OTKA (under T043159) and by INFN Grant TO12.

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